MATC44 Week 2 Notes

1. Pigeon Hole Principle Examples

a) Consider n natural numbers, $x_1, x_2, \ldots, x_n$. Show that there is always a seq of them s.t. their sum is divisible by n.

Soln:
Let the following sums be the pigeons:

$$\left.\begin{array}{l} 1.\ x_1 \\ 2.\ x_1 + x_2 \\ \quad\vdots \\ n.\ x_1 + x_2 + \ldots + x_n \end{array}\right] n \text{ pigeons}$$

By contradiction, suppose that none of the n sums is divisible by n. That means, there are $n-1$ possible remainders left $1, 2, \ldots n-1$. Then, by P.P, 2 of the n sums must give the same remainder when divided by n. This means that the difference of the 2 sums is divisible by n. Since the difference of the 2 sums is itself a sum of seq, there is always a seq of n natural numbers s.t. their sum is divisible by n.

b) Let P be a prime num that is not 2 or 5. Show that among the nums 1, 11, 111, $\underbrace{11\ldots11}_{P\ 1\text{'s}}$, there is always one divisible by P.

Soln:

We have p numbers. By contradiction, assume that none of them are divisible by p. This means there are p-1 possible remainders. By P.P. that means there are at least 2 numbers that will give the same remainder when divided by P. Furthermore, their difference is divisible by P. However, the difference of the 2 nums is always in the form of $(111\ldots1)\cdot 10^k$. Since we know that P cannot be 2 or 5, P cannot divide by 10^k. That means P must be divisible by $11\ldots1$, which contradicts our assumption.
$\therefore$ P must divide one of the numbers.

c) A student studied for 37 days according to the rules:

1. He studied at least 1 hr per day.
2. He never studied more than 12 hrs per day.
3. He studied an integer amount of hrs each day.
4. He studied 60 hrs in total.

Show that there was a period of consecutive days when he studied for exactly 13 hrs.

Soln:

Let A_i be the number of hours studied up to and including the ith day. Then:

1. $A_{i+1} \geq A_i + 1$, bc the student studies at least 1 hr per day.
2. $A_{i+1} \leq A_i + 12$, bc the student studies at most 12 hrs per day.
3. $A_{37} = 60$, bc the student studies 60 hrs in total.
4. $A_i \neq A_j$ for $i \neq j$ (Same reason as 1).

We want to show that there are i, j with $i \geq j+2$ s.t. $A_i = A_j + 13$.
We have the following 2 sets:

1. $\{A_1, A_2, \ldots, A_{37}\}$
2. $\{A_1 + 13, A_2 + 13, \ldots, A_{37} + 13\}$

In total, we have 74 nums
Furthermore, these 74 nums can take at most 73 values.

$1 \leq A_i \leq 60 < 73$
$1 \leq A_i + 13 \leq 73$

By the P.P, there must be 2 nums that have the same value. Hence, for some i,j s.t. $i \geq j+2$, we must have $A_i = A_j + 13$.

d) Let $A_1, \ldots, A_{2000}$ be subsets of the set M s.t. each set A_i contains at least 2/3 of the elements in M. Show that there is an element of M which belongs to at least 1334 of the 2000 subsets A_i.

Soln:
We have 2000 subsets and we want to show that at least 1334 of their elements coincide.
Let $|M|$ be the num of elements in M.
Then, the total num of objects in all 2000 subsets is at least $2000(2/3)(|M|)$. However, all the elements in these subsets are elements of M and hence, there can be at most $|M|$ elements. Thus, we have $(2000)(2/3)(|M|)$ elements taking at most $|M|$ values. By P.P, this means that at least (2000) or 1333.33 elements

that are the same. I.e. This means there is an element belonging to at least 1333.33 sets.

2. Ramsey Theory:

- Version 1: Among 6 ppl there are always 3 who are mutual friends or 3 who are mutual strangers.

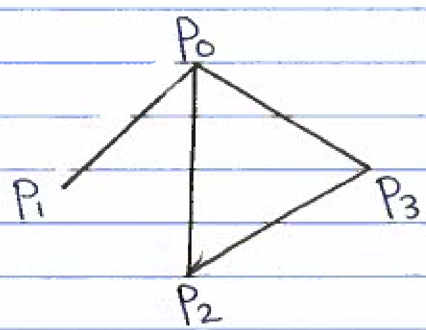
Note: For diagrams, a black line will indicate 2 ppl are friends and a green line will indicate 2 ppl are strangers.

Proof: Consider 1 person, P_0, of the 6 ppl. Then, P_0 is either friends with or strangers with each of the remaining 5 ppl. By P.P, there must be at least 3 of the remaining 5 ppl, P_1, P_2, P_3, s.t. P_0 is either friends with them or strangers with them.

Case 1: Assume that P_0 is friends with P_1, P_2 and P_3.

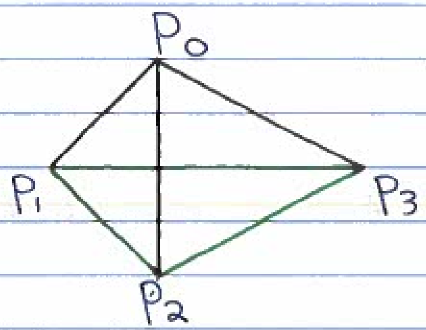
Sub-Case 1: If there is a pair of friends, P_i, P_j for $i,j \in \{1,2,3\}$, among P_1, P_2, P_3, then P_0, P_i and P_j are mutually friends.

E.g.



Sub-Case 2: If there is no pair of friends among P_1, P_2, P_3, then P_1, P_2, P_3 are mutual strangers.

E.g.



Case 2: Assume that P_0 is strangers with P_1, P_2, P_3.

Sub-Case 1: If there is a pair of strangers, P_i, P_j, with $i,j \in \{1,2,3\}$ among P_1, P_2, P_3, then P_0, P_i, P_j are mutual strangers.

Sub-Case 2: If there is no pair of strangers among P_1, P_2, P_3, then P_1, P_2, P_3 are mutual friends.

$\therefore$ No matter how the initial 6 ppl are related to each other, there will always be a group of 3 ppl who are friends or strangers.

- Version 2: Consider any 6 points on the plane and color all edges all edges which connect these 6 points red or blue. Show that there must always be a red triangle or blue triangle.

Proof: Consider one of the 6 points, A. A is connected to 5 more pts, so there are 5 edges which terminate at A. Each of the 5 edges is either red or blue. We have 5 edges (pigeons) and 2 colors (ph's). By P.P, we must have 3 edges that terminate at A which are all red or all blue.

Note: Each graph of 6 pts contains 15 edges, which are either red or blue. Since we have 2 colours, we have 2^{15} or 32k different possible graphs. By the above thm, each of the 32k graphs must contain a red or blue triangle.

- Def of Ramsey Theory: The natural R(m,n) is defined as the smallest natural number that has the following property:

Consider R(m,n) pts on the plane and all edges connecting all pairs of R(m,n) pts. If each edge is either red or blue, then there is always a blue m-gon with all sides and diagonals blue or a red n-gon s.t. all sides/diagonals are red.